SUGGESTED SOLUTION TO HOMEWORK 2

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Problem 1. Prove that for every x in a normed space X, the following identity holds:

$$\|x\| = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, \quad f \neq 0\right\}.$$

Proof. Fix an arbitrary $x_0 \in X$. On the one hand, denote

$$\alpha = \sup\left\{\frac{|f(x_0)|}{\|f\|} : f \in X^*, \quad f \neq 0\right\}.$$

Since for arbitrary $f \in X^*$,

$$|f(x_0)| \leq \|f\| \cdot \|x_0\|,$$

therefore

$$\alpha \leq \|x_0\|.$$

On the other hand, define $\tilde{f}_0$ on $E = \{\alpha x_0 : \alpha \in \mathbb{R}\}$ by

$$\tilde{f}_0(y) = \alpha\|x_0\|, \quad \forall y = \alpha x_0 \in E.$$

Then $\tilde{f}_0$ is a linear functional on E. Moreover,

$$\|\tilde{f}_0(y)\| \leq \|y\|,$$

therefore by Hahn-Banach theorem, there exists a linear functional f_0 on X such that

$$f_0|_E = \tilde{f}_0,$$

and

$$|f_0(x)| \leq \|x\|, \quad \forall x \in X.$$

Then

$$f_0(x_0) = \|x_0\|, \quad \|f_0\| \leq 1,$$

which implies

$$\|x_0\| \leq \alpha$$

□

Problem 2. For $0 < p < 1$, let X be the vector space of all step functions on the interval $[0, 1]$ with the function d defined by

$$d(x, y) = \int_0^1 |x(t) - y(t)|^p dt, \quad \text{for all } x, y \in X.$$

Show that (X, d) is a metric space.

Proof. For arbitrary step functions x on $[0,1]$, we assume there exists $n \in \mathbb{N}$ such that

$$x(t) = \sum_{i=1}^{n} x_i \mathbb{1}_{I_i}(t),$$

where x_i, $i = 1, \cdots, n$, are constants, I_i, $i = 1, \cdots, n$, are mutually disjoint intervals between $[0,1]$ with $\overset{n}{\underset{i=1}{\cup}} I_i = [0,1]$, $\mathbb{1}()_I(t)$ is the indicator function of interval $I \subset [0,1]$,

$$\mathbb{1}_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Therefore it is obvious that X is a vector space. In the following, we prove that $d(\cdot,\cdot)$ is positive definite, symmetric and satisfies the triangle inequality.

Let us prove $d(\cdot,\cdot)$ is positive definite. It is obvious that $d(\cdot,\cdot)$ is positive. Suppose $d(x,y) = 0$, then

$$\begin{aligned}\int_0^1 |x(t) - y(t)|^p dt &= \int_0^1 \left| \sum_{\alpha=1}^{n} x_\alpha \mathbb{1}_{I_\alpha}(t) - \sum_{\beta=1}^{m} y_\beta \mathbb{1}_{J_\beta}(t) \right|^p dt \\ &= \int_0^1 \left| \sum_{\gamma=1}^{l} z_\gamma \mathbb{1}_{k_\gamma}(t) \right|^p dt \\ &= \int_0^1 \sum_{\gamma=1}^{l} |z_\gamma|^p \mathbb{1}_{K_\gamma}(t) dt,\end{aligned}$$

where K_γ, $\gamma = 1, \cdots, l$ are intervals which are obtained by finitely union, intersection and complementation of I_α, $\alpha = 1, \cdots, n$ and J_β, $\beta = 1, \cdots, m$, z_α is the difference between x and y on K_α, therefore for each α,

$$z_\alpha = 0,$$

which implies $x = y$.

We claim that $d(\cdot,\cdot)$ is symmetric. Indeed, for arbitrary step functions x and y,

$$d(x,y) = \int_0^1 |x(t) - y(t)|^p dt = \int_0^1 |y(t) - x(t)|^p dt = d(y,x).$$

We prove that for arbitrary step functions x, y and z, $d(x,y) + d(y,z) \geq d(x,z)$. Since $0 < p < 1$, then $p - 1 < 0$, therefore

$$\begin{aligned}|x(t) - z(t)|^p \leq &(|x(t) - y(t)| + |y(t) - z(t)|)^p \\ \leq &|x(t) - y(t)|(|x(t) - y(t)| + |y(t) - z(t)|)^{p-1} \\ &+ |y(t) - z(t)|(|x(t) - y(t)| + |y(t) - z(t)|)^{p-1} \\ \leq &|x(t) - y(t)|^p + |y(t) - z(t)|^p,\end{aligned}$$

which implies that

$$\int_0^1 |x(t) - y(t)|^p dt + \int_0^1 |y(t) - z(t)|^p dt \geq \int_0^1 |x(t) - z(t)|^p dt.$$

□

Problem 3. Show that $p(x) = \limsup x_n$, where $x \in \ell_\infty$, $x(n) \in \mathbb{R}$, defines a sublinear functional on ℓ_∞.

Proof. It is obvious that $p(x)$ is positive homogeneous. We claim that $p(x)$ is also subadditive. Indeed, for arbitrary $x, y \in \ell_\infty$, we have $x + y \in \ell_\infty$. Since

$$\sup_{m \geq n} (x+y)(m) \leq \sup_{m \geq n} x(m) + \sup_{m \geq n} y(m),$$

by taking n goes to infinity,

$$p(x+y) \leq p(x) + p(m).$$

$\square$

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